$$|y_2'(\lambda, \eta, x) + i|\beta|e^{-i|\beta|x}| \leqslant |\beta|\left[\exp\left(\eta\int_x^\infty B_2\,dt\right) - 1\right]$$

Using inequalities (20) and (21), we obtain from the definition of $\psi_1(\lambda, \eta, x)$

$$|\psi_1(\lambda, \eta, 0) + |\beta|(1-i)| \leqslant |\beta|\left\{\exp\left[\eta\int_0^\infty (A_1 + B_2)\,dt\right] + \exp\left[\eta\int_0^\infty (A_2 + B_1)\,dt\right] - 2\right\}$$

It is seen from the latter formula that $\psi_1(\lambda, \eta, 0) \neq 0$ follows from inequality (19). The first inequality of (21) is satisfied not only in the circle O but also in the interval J. Consequently $y_1'(\lambda, \eta, 0) \neq 0$, follows from the inequality (19), i.e., (11) is not satisfied, meaning, there are no eigenvalues in this interval. The theorem is proved.

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PMM U.S.S.R., Vol.52, No.5, pp.653-659, 1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

AXISYMMETRIC FLEXURAL OSCILLATIONS OF A THIN DISC*

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Using methods of the theory of singular perturbations /1-3/, we construct the asymptotic forms of the eigenfrequencies of flexural low-frequency oscillations of a thin disc. Application of the method of homogeneous solutions /4/ or the superposition method /5/ reduces the problem under consideration to an infinite system of linear algebraic equations. Unlike these approaches, the theory of singular perturbations enables us to obtain explicit formulae for corrections to the oscillation eigenfrequencies obtained from the classical theory of plates.

1. Formulation of the problem. We consider the problem of the axially-symmetric flexural oscillations of a thin disc of radius a and thickness $2h$ ($\varepsilon = h/a \ll 1$) in a system of cylindrical coordinates (r, φ, z). The planes $z = \pm h$ and the side surface $r = a$ are free from stresses.

In dimensionless coordinates $\rho = r/a$, $\xi = z/h$ the problem may be written in the form

$$(1-2\nu)\partial_\xi^2 u_r + \varepsilon\partial_\rho\partial_\xi u_z + 2(1-\nu)\varepsilon^2\partial_\rho(\rho^{-1}\partial_\rho(\rho u_r)) + \mu u_r = 0 \tag{1.1}$$

$$2(1-\nu)\partial_\xi^2 u_z + \varepsilon\rho^{-1}\partial_\rho(\rho\partial_\xi u_r) + (1-2\nu)\varepsilon^2\Delta u_z + \mu u_z = 0$$

$$G(\partial_\xi u_r + \partial_\rho u_z)_{\xi=\pm1} = 0 \tag{1.2}$$

$$d\,[2(1-\nu)\partial_\xi u_z + 2\nu\rho^{-1}\partial_\rho(\rho u_r)]_{\xi=\pm1} = 0$$

$$d\,[2(1-\nu)\partial_\rho u_r + 2\nu(\partial_\xi u_z + \rho^{-1}u_r)]_{\rho=1} = 0 \tag{1.3}$$

$$G(\partial_\xi u_r + \partial_\rho u_z)_{\rho=1} = 0$$

**Prikl.Matem.Mekhan.*, 52, 5, 837-843, 1988

$$d = G/(1-2\nu),\ \ \mu = \rho_1 h^2\omega^2/d,\ \ \Delta = \partial_\rho{}^2 + \rho^{-1}\partial_\rho$$

where $u_r(\rho, \xi, \varepsilon)$, $u_z(\rho, \xi, \varepsilon)$ are the coordinates of the displacement vector, G is the shear modulus, ν is Poisson's ratio, ρ_1 is the density, and ω is the oscillation frequency. We also introduce the dimensionless coordinate $\tau = (\rho - 1)/\varepsilon$.

We will seek an asymptotic solution of (1.1)-(1.3) in the form of the sum of a regular solution $\mathbf{v}(\rho, \xi, \varepsilon)$ and a boundary-layer type solution $\mathbf{w}(\tau, \xi, \varepsilon)$

$$\mathbf{u} = h(\mathbf{v}(\rho, \xi, \varepsilon) + \mathbf{w}(\tau, \xi, \varepsilon)) \tag{1.4}$$

$$\mathbf{v}(\rho, \xi, \varepsilon) = \sum_{n=0}^{N} \varepsilon^n \mathbf{v}^{(n)}(\rho, \xi) \tag{1.5}$$

$$\mathbf{w}(\tau, \xi, \varepsilon) = \varepsilon^m \sum_{n=0}^{N-m} \varepsilon^n \mathbf{w}^{(n)}(\tau, \xi)$$

$$\mu = \varepsilon^4 \sum_{n=0}^{N-4} \mu_n \varepsilon^n$$

Each of the functions $\mathbf{v}$ and $\mathbf{w}$ is, by construction, an asymptotic solution up to $O(\varepsilon^{N+1})$ of (1.1) and (1.2), $\mathbf{w}$ tends exponentially to zero as $\tau \to -\infty$, and (1.4) satisfies (1.3) up to $O(\varepsilon^{N+1})$.

2. **Construction of the internal solution.** Substituting the asymptotic expansion (1.5) for $\mathbf{v}$ and μ into (1.1), (1.2) and grouping together terms with the same powers of ε we obtain the following boundary-value problem for the functions $v_r^{(k)}$ and $v_z^{(k)}$ $(0 \leqslant k \leqslant N)$:

$$(1-2\nu)\,\partial_\xi{}^2 v_r^{(k)} + \partial_\rho\partial_\xi v_z^{(k-1)} + P_r^{(k-2)}(\rho, \xi) = 0 \tag{2.1}$$

$$(\partial_\xi v_r^{(k)} + \partial_\rho v_z^{(k-1)})_{\xi=\pm 1} = 0$$

$$2(1-\nu)\,\partial_\xi{}^2 v_z^{(k)} + \rho^{-1}\partial_\rho(\rho\partial_\xi v_r^{(k-1)}) + P_z^{(k-2)}(\rho, \xi) = 0 \tag{2.2}$$

$$[2(1-\nu)\,\partial_\xi v_z^{(k)} + 2\nu\rho^{-1}\partial_\rho(\rho v_r^{(k-1)})]_{\xi=\pm 1} = 0$$

$$P_r^{(k-2)}(\rho, \xi) = 2(1-\nu)\,\partial_\rho(\rho^{-1}\partial_\rho(\rho v_r^{(k-2)})) + \sum_{n=0}^{k-4} \mu_n v_r^{(k-n-4)}$$

$$P_z^{(k-2)}(\rho, \xi) = (1-2\nu)\,\Delta v_z^{(k-2)} + \sum_{n=0}^{k-4} \mu_n v_z^{(k-n-4)}$$

Here and in the future, all quantities with negative indeces are taken to be equal to zero. Moreover, the expression joined by the summation sign with summation from $n = i$ to $n = j$ with $j < i$ is also equal to zero.

We are considering flexural oscillations of a disc, and so

$$v_r^{(k)}(\rho, -\xi) = -v_r^{(k)}(\rho, \xi),\ \ v_z^{(k)}(\rho, -\xi) = v_z^{(k)}(\rho, \xi)$$

In this case, the solution to (2.1) may be written in the form

$$v_r^{(k)}(\rho, \xi) = \frac{1}{1-2\nu}\Big[-\int_0^\xi (\partial_\rho v_z^{(k-1)}(\rho, \eta) + (\xi - \eta)\,P_r^{(k-2)}(\rho, \eta))\,d\eta + 2\nu\xi\partial_\rho v_z^{(k-1)}(\rho, 1) + \xi\int_0^1 P_r^{(k-2)}(\rho, \eta)\,d\eta\Big] \tag{2.3}$$

Given that the solubility condition

$$\frac{1-2\nu}{\rho}\,\partial_\rho(\rho v_r^{(k-1)}(\rho, 1)) + \int_0^1 P_z^{(k-2)}(\rho, \xi)\,d\xi = 0 \tag{2.4}$$

is satisfied, the solution of (2.2) is described by the formula

$$v_z^{(k)}(\rho, \xi) = f_k(\rho) - \frac{1}{2(1-\nu)}\int_0^\xi \Big[\frac{1}{\rho}\,\partial_\rho(\rho v_r^{(k-1)}(\rho, \eta)) + (\xi - \eta)\,P_z^{(k-2)}(\rho, \eta)\Big]\,d\eta \tag{2.5}$$

where $f_k(\rho)$ is an arbitrary function that does not depend on ξ.

From (2.3), (2.5) we determine by integration

$$v_r^{(k)}(\rho, \xi) = \xi f'_{k-1}(\rho) + (1-\nu)^{-1}\,({}^1/_6\,(2-\nu)\xi^3 - \xi)\partial_\rho\Delta f_{k-3} + V_r^{(k,\,k-5)}(\rho, \xi) \tag{2.6}$$

$$v_z^{(k)}(\rho, \xi) = f_k(\rho) + {}^1/_2 \nu \xi^2 (1-\nu)^{-1} \Delta f_{k-2} + V_z^{(k,\, k-4)}(\rho, \xi)$$

The functions $\mathbf{V}^{(k,\,n)}$ depend on $f_i(\rho)$ when $i \leqslant n$; when $n < 0$ $\mathbf{V}^{(k,\,n)} = 0$.

The solubility condition (2.4), taking account of (2.5) and (2.6), reduces to an equation for $f_l(\rho)$ $(l = k - 4)$

$$\Delta^2 f_l = c_1 M_l(\rho) + G_{l,\, l-2}(\rho) \tag{2.7}$$

$$c_1 = \frac{3(1-\nu)}{2(1-2\nu)}, \quad M_l(\rho) = \sum_{n=0}^{l} \mu_n f_{l-n}(\rho)$$

The functions $G_{l,\,j}(\rho)$ depend on $f_n(\rho)$ when $n \leqslant j$; when $j < 0$ $G_{l,\,j} = 0$.

Continuing the calculations, we find

$$\begin{aligned} V_r^{(k,\,k-5)}(\rho, \xi) = {} & \frac{1}{240(1-\nu)(1-2\nu)} \Big\{ [-3(1-\nu)(3-\nu)\xi^5 + \\ & 50(3-2\nu)\xi^3 - 60(7-2\nu)\xi]\, M'_{k-5} + \\ & \frac{1}{1-\nu} [{}^1/_{14}(1-\nu)^2(4-\nu)\xi^7 - {}^1/_5(7\nu^3 - 10\nu^2 - 26\nu + 24)\xi^5 + \\ & (43 - 30\nu - 8\nu^2)\xi^3 + 4(2\nu^2 + 15\nu - 27)\xi]\, \partial_\rho \Delta M_{k-7} \Big\} + \\ & V_r^{(k,\,k-9)}(\rho, \xi) \end{aligned} \tag{2.8}$$

$$\begin{aligned} V_z^{(k,\,k-4)}(\rho, \xi) = {} & \frac{\xi^2}{16(1-\nu)(1-2\nu)} \Big\{ [2(1+4\nu) - (1-\nu^2)\xi^2]\, M_{k-4} + \\ & \frac{1}{30(1-\nu)} [(1-\nu)^2(2+\nu)\xi^4 + 2(7\nu^3 + 3\nu^2 - 2\nu - 3)\xi^2 + \\ & 6(1+4\nu)]\, \Delta M_{k-6} \Big\} + V_z^{(k,\,k-8)}(\rho, \xi) \end{aligned}$$

$$G_{l,\,l-2}(\rho) = c_2 \Delta M_{l-2}(\rho) + c_3 \mu_0^2 f_{l-4} + G_{l,\,l-6}(\rho)$$

$$c_2 = \frac{7\nu - 17}{10(1-2\nu)}, \quad c_3 = \frac{33\nu^2 + 424\nu - 422}{700(1-2\nu)^2}$$

The force vector $\mathbf{F}(\xi, \varepsilon)$ on the side surface, corresponding to the displacement $\boldsymbol{v}(\rho, \xi, \varepsilon)$ is found from the formulae

$$\mathbf{F}(\xi, \varepsilon) = d \sum_{k=0}^{N} \varepsilon^k \mathbf{F}^{(k)}(\xi) \tag{2.9}$$

$$F_r^{(k)}(\xi) = [2(1-\nu)\, \partial_\rho v_r^{(k-1)} + 2\nu(\partial_\xi v_z^{(k)} + \rho^{-1} v_r^{(k-1)})]_{\rho=1}$$

$$F_z^{(k)}(\xi) = (1-2\nu)(\partial_\xi v_r^{(k)} + \partial_\rho v_z^{(k-1)})_{\rho=1}$$

Substituting (2.6)-(2.8) we obtain the formulae

$$\begin{aligned} F_r^{(k)}(\xi) = {} & -\frac{3\xi}{c_1} \Big[f''_{k-2} + \nu f'_{k-2} + \Big(\frac{2-\nu}{6}\xi^2 - 1 \Big) \partial_\rho \Delta f_{k-4} \Big]_{\rho=1} + \\ & \Big(\xi^3 - \frac{6-\nu}{2(1-\nu)}\xi \Big) M_{k-4}(1) + \\ & \frac{1}{120(1-\nu)^2} \{ [-9(1-\nu)^2\xi^5 + 2(7 + 6\nu - 23\nu^2)\xi^3 - \\ & 6(1+4\nu)(2-\nu)\xi]\, \Delta M_{k-6} + (1-\nu)[3(1-\nu)(3-\nu)\xi^5 - \\ & 50(3-2\nu)\xi^3 + 60(7-2\nu)\xi]\, M'_{k-5} \}_{\rho=1} + F_r^{(k,\,k-8)}(\xi) \end{aligned} \tag{2.10}$$

$$\begin{aligned} F_z^{(k)}(\xi) = {} & \frac{1-\xi^2}{4} \Big\{ -\frac{6}{c_1} \partial_\rho \Delta f_{k-3} + \Big(\xi^2 - \frac{7-2\nu}{1-\nu} \Big) M'_{k-5} - \\ & \frac{1}{60(1-\nu)^2} [3(1-\nu)^2\xi^4 + 2(8\nu^2 + 9\nu - 12)\xi^2 + \\ & 4(27 - 15\nu - 2\nu^2)]\, \partial_\rho \Delta M_{k-7} \Big\}_{\rho=1} + F_z^{(k,\,k-9)}(\xi) \end{aligned}$$

The functions $F^{(k,\,n)}$ depend on $f_l(\rho)$ for $l \leqslant n$. We can satisfy the boundary conditions (1.3) up to $O(\varepsilon^4)$, requiring that the following conditions for $f_n(\rho)$ should be satisfied with $n = 0$ and $n = 1$

$$(f_n'' + \nu f_n')_{\rho=1} = \partial_\rho(\Delta f_n)|_{\rho=1} = 0 \tag{2.11}$$

Consequently, in expansion (1.4) for $\mathbf{w}$ we can set $m = 4$.

The general solution of (2.7) and (2.11) for $f_0(\rho)$ takes the form

$$f_0(\rho) = \alpha_0 (I(p\rho) - J(p\rho)), \quad p = (c_1 \mu_0)^{1/4} \tag{2.12}$$

$$J(p\rho) = J_0(p\rho)/J_1(p), \quad I(p\rho) = I_0(p\rho)/I_1(p)$$

where α_0 is an arbitrary function, J_0 and J_1 are Bessel functions, I_0 and I_1 are modified Bessel functions, and the number p is one of the positive roots of the equation

$$p\,(J\,(p) + I\,(p)) = 2\,(1 - \nu) \tag{2.13}$$

To make the following calculations more convenient, we will define the constant α_0 from the normalization condition

$$\int_0^1 \rho f_0^2(\rho)\,d\rho = 1$$

whence we find

$$\alpha_0 = \sqrt{2}\,[I^2(p) + J^2(p) - 2p^{-1}(I(p) + J(p))]^{-1/2}$$

Problem (2.7), (2.11) for $f_1(\rho)$ is soluble when the condition $\mu_1 = 0$ is satisfied. Here the problems for f_0 and f_1 are identical, and we can set $f_1 = 0$.

To find the functions $f_n(\rho)$, and the numbers μ_n with $n > 1$ it is necessary to consider the boundary-layer solution **w**.

3. Construction of the boundary-layer solution. We denote by $\sigma(\tau, \xi, \varepsilon)$ the stress tensor corresponding to the displacements **w**. Using the asymptotic expansion (1.5) of the vector function **w**, we obtain the following formulae for the components of the asymptotic form of the tensor σ:

$$\sigma(\tau, \xi, \varepsilon) = \varepsilon^4 d \sum_{n=0}^{N-4} \varepsilon^n \sigma^{(n)}(\tau, \xi) \tag{3.1}$$

$$\sigma_{rr}^{(k)}(\tau, \xi) = \sigma_{rr}^{(k,0)} + 2\nu T_{k-1},\quad \sigma_{zz}^{(k)}(\tau, \xi) = \sigma_{zz}^{(k,0)} + 2\nu T_{k-1}$$

$$\sigma_{rr}^{(k,0)}(\tau, \xi) = 2\,(1-\nu)\partial_\tau w_r^{(k)} + 2\nu\partial_\xi w_z^{(k)}$$

$$\sigma_{zz}^{(k,0)}(\tau, \xi) = 2\,(1-\nu)\partial_\xi w_z^{(k)} + 2\nu\partial_\tau w_r^{(k)}$$

$$\sigma_{rz}^{(k)}(\tau, \xi) = (1-2\nu)(\partial_\xi w_r^{(k)} + \partial_\tau w_z^{(k)})$$

$$\sigma_{\varphi\varphi}^{(k)}(\tau, \xi) = 2\nu\,(\partial_\tau w_r^{(k)} + \partial_\xi w_z^{(k)}) + 2\,(1-\nu)\,T_{k-1}$$

$$\sigma_{r\varphi}^{(k)} = \sigma_{z\varphi}^{(k)} = 0$$

$$T_k(\tau, \xi) = \sum_{n=0}^{k} w_r^{(n)}(\tau, \xi)(-\tau)^{k-n}$$

The equations for the components of the tensor $\sigma^{(k)}$ in the semi-infinite strip $D = \{\tau < 0,\ |\xi| < 1\}$ are reduced to the form

$$\partial_\tau\sigma_{rr}^{(k,0)} + \partial_\xi\sigma_{rz}^{(k)} = Q_r^{(k-1)} \tag{3.2}$$

$$\partial_\tau\sigma_{rz}^{(k)} + \partial_\xi\sigma_{zz}^{(k,0)} = Q_z^{(k-1)}$$

$$Q_r^{(k)}(\tau, \xi) = -2\,\partial_\tau T_k - \sum_{n=0}^{k}(-\tau)^{k-n}(\sigma_{rr}^{(n)} - \sigma_{\varphi\varphi}^{(n)}) - S_r^{(k-3)}$$

$$Q_z^{(k)}(\tau, \xi) = -2\nu\partial_\xi T_k - \sum_{n=0}^{k}(-\tau)^{k-n}\sigma_{rz}^{(n)} - S_z^{(k-3)}$$

$$\mathbf{S}^{(k)}(\tau, \xi) = \sum_{n=0}^{k}\mu_n \mathbf{w}^{(k-n)}$$

On the sides $\xi = \pm 1$ and $\tau = 0$ we are given the following boundary conditions:

$$\sigma_{rz}^{(k)}(\tau, \pm 1) = 0,\quad \sigma_{zz}^{(k,0)}(\tau, \pm 1) = -2\nu T_{k-1}(\tau, \pm 1) \tag{3.3}$$

$$\sigma_{rr}^{(k,0)}(0, \xi) = -2\nu T_{k-1}(0, \xi) - F_r^{(k+4)}(\xi) \tag{3.4}$$

$$\sigma_{rz}^{(k)}(0, \xi) = -F_r^{(k+4)}(\xi)$$

Problems (3.2)-(3.4) are boundary-value problems of two-dimensional elasticity theory on the bending of a semi-infinite strip D. The conditions for the existence of exponentially decaying solutions of these problems were studied in /6-9/. These conditions may be written in the form

$$\langle F_z^{(k+4)}(\xi)\rangle = -2\nu\int_{-\infty}^{0} T_{k-1}(\tau, \xi)\,d\tau\,\Big|_{\xi=-1}^{1} - \iint Q_z^{(k-1)}(\tau, \xi)\,d\tau\,d\xi \tag{3.5}$$

$$\langle F_r^{(k+4)}(\xi)\,\xi\rangle = \iint(\tau Q_z^{(k-1)} - \xi Q_r^{(k-1)})\,d\tau\,d\xi - 2\nu\,\langle \xi T_{k-1}(0, \xi)\rangle + 2\nu\int_{-\infty}^{0} T_{k-1}(\tau, \xi)\,\tau\,d\tau\,\Big|_{\xi=-1}^{1} \tag{3.6}$$

Here and henceforth angular brackets signify integration with respect to ξ from $\xi = -1$ to $\xi = 1$, and the double integrals are evaluated over the region D.

We multiply the second equation of (3.2) by τ and integrate over D. After integration by parts taking account of (3.3) we obtain

$$\iint (\sigma_{rz}^{(k)} + Q_z^{(k-1)}\tau)\, d\tau\, d\xi = -2\nu \int_{-\infty}^{0} \tau T_{k-1}(\tau, \xi)\, d\tau \Big|_{\xi=-1}^{1}$$

Substituting the expression for $Q_z^{(k-1)}$ into this formula, we arrive at the relationship

$$\sum_{n=0}^{k} \iint (-\tau)^{k-n} \sigma_{rz}^{(n)}\, d\tau\, d\xi = \iint S_z^{(k-4)} \tau\, d\tau\, d\xi$$

Then (3.5) is transformed into

$$\langle F_z^{(k+4)}(\xi)\rangle = \iint (S_z^{(k-4)} + \tau S_z^{(k-5)})\, d\tau\, d\xi$$

Hence, for $k < 4$ we obtain a boundary condition for $f_l(\rho)$ $(l = k + 1 \leqslant 4)$ of the form

$$(\partial_\rho \Delta f_l)_{\rho=1} = -\frac{34 - 9\nu}{20(1 - 2\nu)} M'_{l-2}(1) \tag{3.7}$$

Condition (3.6) may be written in the form

$$\langle \xi F_r^{(k+4)}(\xi)\rangle = -\sum_{n=0}^{k-1} \iint (-\tau)^{k-n-1} (\tau\sigma_{rz}^{(n)} - \xi\sigma_{rr}^{(n)} + \xi\sigma_{\varphi\varphi}^{(n)})\, d\tau\, d\xi + \iint (\xi S_r^{(k-4)} - \tau S_z^{(k-4)})\, d\tau\, d\xi \tag{3.8}$$

For $k = 0$, the right-hand sdie of (3.8) is equal to zero. From (3.1)-(3.4) we obtain the equations

$$\sigma_{\varphi\varphi}^{(n)} = \nu(\sigma_{rr}^{(n)} + \sigma_{zz}^{(n)}) + 2(1 + \nu)(1 - 2\nu) T_{n-1} \quad (n \geqslant 0)$$

$$\iint \tau^l \sigma_{rz}^{(0)}\, d\tau\, d\xi = \iint \tau^l \xi \sigma_{rr}^{(0)}\, d\tau\, d\xi = 0 \quad (l \geqslant 0)$$

$$2\iint \xi\sigma_{zz}^{(0)}\, d\tau\, d\xi + \langle \xi^2 F_z^{(4)}(\xi)\rangle = 0$$

(the proof is carried out by integrating (3.2) by parts with appropriate multipliers). It follows from these formulae that with $k = 1$ conditions (3.8) reduce to the form

$$\langle \xi F_r^{(k+4)}(\xi) - {}^1/_2 \nu \xi^2 F_z^{(k+3)}(\xi)\rangle = 0 \tag{3.9}$$

Conditions (3.8) with $k = 2$ also reduce to the form (3.9) using analogous, but more complicated, calculations. From (3.9) and (2.10) we obtain the second boundary condition for the functions $f_l(\rho)$ with $l = k + 2 \leqslant 4$

$$(f_l'' + \nu f_l')_{\rho=1} = \frac{1}{20(1 - 2\nu)} \Big\{4(4 + \nu)(1 - 2\nu)\partial_\rho \Delta f_{l-2} - (24 + \nu) M_{l-2} - \frac{87(\nu^2 + 1) + 316\nu}{140(1 - \nu)} \mu_0 \Delta f_{l-4} + \frac{779 - 60\nu - 19\nu^2}{28} \mu_0 f_{l-4}'\Big\}_{\rho=1} \tag{3.10}$$

4. Calculation of the asymptotic forms of the eigenvalues.

We consider the boundary-value problem (2.7), (3.7), (3.10) for the functions $f_l(\rho)$ with $l > 1$. The solutions $f_l(\rho)$ are determined apart from the term $\alpha_l f_0$ (the solution of the homogeneous problem) and so it is convenient to determine the constant α_l from the supplementary orthogonality condition

$$\int_0^1 \rho f_0(\rho) f_l(\rho)\, d\rho = 0 \tag{4.1}$$

We multiply (2.7) by $\rho f_0(\rho)$ and integrate with respect to ρ from 0 to 1. After integration by parts, taking account of (4.1) we arrive at a formula that determines the quantity

μ_l

$$c_1\mu_l = -\int_0^1 \rho f_0(\rho) G_{l,\,l-2}(\rho)\,d\rho + [\partial_\rho(\Delta f_l) f_0 - (f_l'' + \nu f_l') f_0']_{\rho=1} \tag{4.2}$$

With $l \leqslant 3$, in particular, we obtain

$$c_1\mu_2 = \mu_0 p c_2 \alpha_0^2 [p + 7(1-\nu)^2 (I(p) - J(p))/(7\nu - 17)], \quad \mu_3 = 0 \tag{4.3}$$

The solutions of the boundary-value problems for $f_2(\rho)$ and $f_3(\rho)$ have the form

$$f_2(\rho) = a_2 J(p\rho) + b_2 I(p\rho) + k_-\rho J_1(p\rho)/J_1(p) + k_+\rho I_1(p\rho)/I_1(p) + \alpha_2 f_0(\rho), \quad f_3(\rho) = 0$$

$$k_\pm = \alpha_0 (c_1\mu_2 \pm c_2\mu_0 p^2)/(4p^3)$$

$$a_2 = k_-(2/p + J(p)) + \delta, \quad b_2 = -k_+(2/p + I(p)) + \delta$$

$$\delta = \frac{9\nu - 34}{20(1-2\nu)p^2}\alpha_0\mu_0$$

From (4.1), we determine

$$\alpha_2 = \alpha_0 [(b_2 - a_2)(I(p) + J(p))/p + a_2(1 + J^2(p)) + b_2(1 - I^2(p)) + (k_- + k_+)(pI(p)J(p) - I(p) - J(p))/p^2]/2$$

From (4.2) with $l = 4$ we find.

$$c_1\mu_4 = {}^1/_{40}(1-2\nu)^{-1}\{2\mu_0[(24+\nu) f_0' f_2 - (34 - 9\nu) f_0 f_2'] - 20(1-\nu)\mu_2 f_0 f_0' + \nu(8-\nu)\mu_0 f_0'^2\}_{\rho=1} - c_3\mu_0^2 - c_2\int_0^1 \rho f_0(\mu_0\Delta f_2 + \mu_2\Delta f_0)\,d\rho$$

Table 1 gives the values of μ_0, μ_2 and μ_4 for the first six oscilation frequencies with $\nu = {}^1/_3$. The formulae obtained enable us to find more exact values for the eigenfrequencies the following are satisfied:

$$|\varepsilon^2\mu_2/\mu_0| < 1, \quad |\varepsilon^2\mu_4/\mu_2| < 1$$

whence $\varepsilon < \min(\sqrt{|\mu_0/\mu_2|}, \sqrt{|\mu_2/\mu_4|})$. For the first oscillation frequency, we apply the method with $\varepsilon < {}^1/_4$, and for the second, we apply it with $\varepsilon < {}^1/_8$.

Table 1

№	μ_0	μ_2	μ_4
1	$0.275\cdot10^2$	$-0.338\cdot10^3$	$0.520\cdot10^4$
2	$0.494\cdot10^3$	$-0.275\cdot10^5$	$0.182\cdot10^7$
3	$0.257\cdot10^4$	$-0.328\cdot10^6$	$0.497\cdot10^8$
4	$0.820\cdot10^4$	$-0.188\cdot10^7$	$0.508\cdot10^9$
5	$0.201\cdot10^5$	$-0.721\cdot10^7$	$0.307\cdot10^{10}$
6	$0.418\cdot10^5$	$-0.216\cdot10^8$	$0.133\cdot10^{11}$

Table 2

№	$N=4$	6	8	[10]
1	0.1815	0.1811	0.1811	0.1811
2	0.7703	0.7617	0.7619	0.7615
3	1.756	1.711	1.714	1.711
4	3.138	2.991	3.007	2.999
5	4.914	4.548	4.612	4.587
6	7.085	6.310	6.509	6.438

Table 2 gives the values of $\Omega = \omega l\sqrt{\rho_1/G}$ for $\nu = {}^1/_3$, $\varepsilon = 0.02$ for the first six oscillation eigenfrequencies; in the last column, we give the values of Ω according to refined Mindlin theory /10/.

The values of Ω with $N = 4$ correspond to those found by the classical theory of plates. With $N = 6$ and $N = 8$ we obtain more exact values of Ω.

Remarks. 1°. In an analogous way, we can construct the asymptotic solution for the case of oscillations that are symmetric about the central plane, and also with other boundary conditions on the side surface of the disc.

2°. The boundary conditions (3.7), (3.10) for the functions $f_l(\rho)$ may be obtained from the results of /11/. It should be noted, however, that the method described above enables us to construct an asymptotic solution in the case of a plate of arbitrary form /3/ and variable thickness, where the results of /11/ cannot be applied.

The author thanks V.V. Kucherenko for suggesting the problem and for discussing the results.

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Translated by H.T.

PMM U.S.S.R.,Vol.52,No.5,pp.659-664,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

ANTIPLANE DYNAMICAL CONTACT PROBLEM FOR AN ELECTROELASTIC LAYER*

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The antiplane dynamic contact problem of the excitation of a semibounded electroelastic layer with a lower boundary sharply constricted by a single electrode as the simplest transformer of electroelastic waves is considered. The electrode is modelled by an absolutely rigid polar stamp. In the region of contact between the electrode and the medium, the electric potential and the amplitudes of the shear displacements are given, and outside this region the surface is free from stress and normal component of the magnetic induction is equal to zero.

One of the approaches to studying the propagation laws for electroelastic shear waves in a medium and on a surface, where this approach is based on the use of the method of fictitious absorption is proposed. A comparative analysis of the behaviour of the basic characteristics of the problem for the coupled and uncoupled problems is given, and the behaviour of the amplitude-frequency dependence on the electrode width and the oscillation frequency is studied.

1. Let the medium occupy the region $-\infty \leqslant x, z \leqslant \infty, 0 \leqslant y \leqslant h$. As an electroelastic material, we consider an *XY*-cut of piezoelectric crystals of the 6*mm* hexagonal crystal symmetry class and a piezoelectric ceramic polarized along the z-axis. This case corresponds to the excitation of a shear surface waves $w_0(x, y)e^{-i\omega t}$.

The propagation of electroelastic shear waves in the quasistatic approximation for the

**Prikl.Matem.Mekhan.*,52,5,844-849,1988